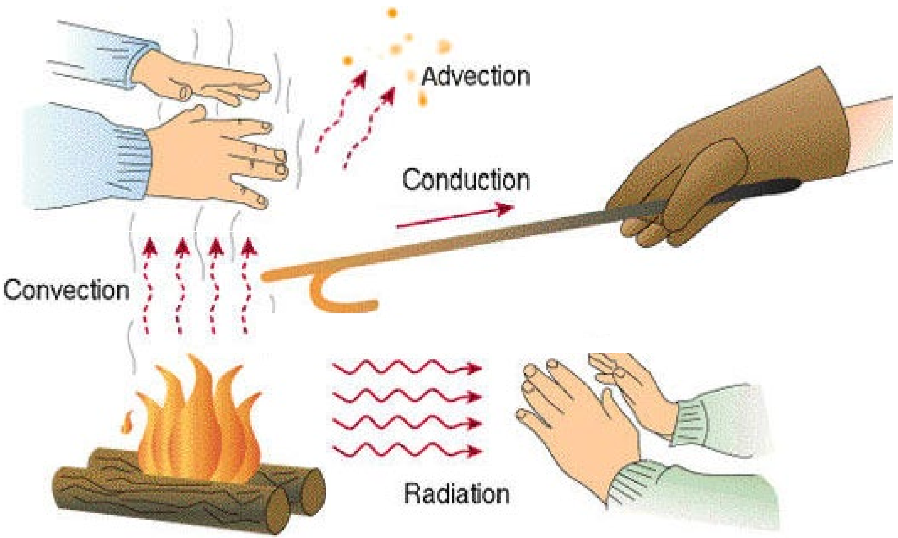
Contents

3. Physics of Microfluidic Systems

3.5 Basic Modes

- Radiation
  - Transport of heat through
    - Vacuum
    - Transparent medium
- Convection
  - Bulk flow carries heat
  - Modes
    - Forced ("pumping")
    - Free ("buoyancy")



- Conduction (also: heat diffusion)
  - Objects that are in physical contact
  - Direct microscopic exchange of kinetic energy of particles
- Advection
  - Transport mechanism of a fluid
  - Depending on motion and momentum of fluid

3.5. Heat Transport

1. **Dynamics of Heat Transport**
2. Boussinesq Approximation
3. Thermal Boundary Conditions
4. Heat Transfer
5. Newton‘s Law of Cooling
6. Dimensionless Parameters
7. Forced-Convection Heat Transfer
8. Heat Transfer in Microchannels

3.5.1. Dynamics of Heat Transport

- Velocity field $\boldsymbol{v}(\boldsymbol{r},t)$: Conservation of
  - **Mass** $$\nabla \cdot \boldsymbol{v} = 0$$
  - **Momentum (Navier Stokes)** $$\varrho_\infty \left[\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] = -\nabla p + \eta \nabla^2 \boldsymbol{v} + \varrho_\infty \boldsymbol{g}$$

- Temperature distribution $T(\boldsymbol{r},t)$: Conservation of (thermal) energy
  - **Convection (directed)**
  - **Diffusion (isotropic)**
  - **Heat source / drain (e.g., boundary)**

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \lambda_D \Delta T + Q'''$$

- Convection-diffusion equation
- Set of three DEs coupling $\boldsymbol{v}(\boldsymbol{r},\boldsymbol{t})$ and $\boldsymbol{T}(\boldsymbol{r},\boldsymbol{t})$

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3.5.2. Boussinesq Approximation

- Very complex system of DEs
  - Simplification needed
- Model
  - Principal impact of temperature through ΔT
  - Density profile $\rho(T)$
- Combination with NS
  - Density ρ varies in **buoyancy-related term** only

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\varrho_\infty \left[\frac{\partial}{\partial t} \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \right] = -\nabla p + \eta \nabla^2 \boldsymbol{v} + \varrho_\infty \boldsymbol{g}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \lambda_D \Delta T + Q'''$$

$$\varrho_\infty \left[\frac{\partial}{\partial t} \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \right] = -\nabla p + \eta \nabla^2 \boldsymbol{v} + \varrho(T) \boldsymbol{g}$$

  - Linear relation for mass expansion

$$\varrho(T) = \varrho_0 (1 - \alpha_V [T(\boldsymbol{x}, t) - T_0])$$

3.5.2. Boussinesq Approximation

$$\varrho_\infty \left[\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p + \eta\nabla^2\boldsymbol{v} + \varrho_0(1-\alpha_V[T(\boldsymbol{x},t)-T_0])\boldsymbol{g}$$

- Simplified coupling between momentum and energy equations
- Estimate for thermal buoyancy force
  - Isothermal fluid at $T = T_0$
  - Wall kept at $T = T_{\text{heat}}$

$$|\boldsymbol{f}_g| = \varrho\|\boldsymbol{g}\|\alpha_V(T_{\text{heat}} - T_0)$$

- Example
  - $\alpha_V = 10^{-3}\ \text{K}^{-1}$
  - $\Delta T = 10\ \text{K}$
  - $f_g = 10^2\ \text{N m}^{-3}$

Comparison: Buoyancy of gas bubbles in water

$$f_{\Delta\varrho} = \Delta\varrho g \approx 10^4\,\text{N m}^{-3}$$

"Bubbles rise faster than warm liquid."

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3.5.3. Thermal Boundary Conditions

- Two disjoint volume regions Ω_1 and Ω_2
- Overall boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$

- Dirichlet
  - Temperature specified on boundary region
  - Walls thermalized by reservoir at $T = \text{const.}$

$$T|_{\partial\Omega_1} = T_1$$

- von Neumann
  - Heat flux j_Q induced by T-gradient

$$-\lambda \frac{\partial T}{\partial n}|_{\partial\Omega_2} = j_Q$$

- $j_Q = 0$
  - No exchange of energy with environment
  - Adiabatic

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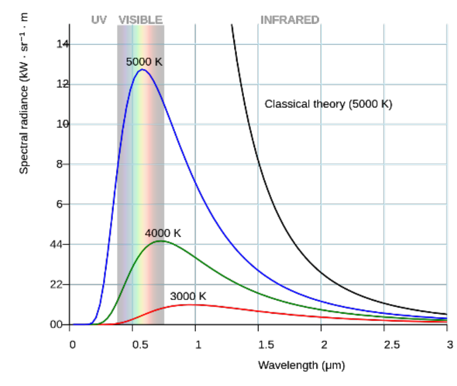
3.5.4. Radiative Transfer

- Mode of heat transfer
  - No heat diffusion (thermal conduction) between solid and fluid
  - No convective heat transfer
- Mechanism: Radiation
  - QM: black body radiation
  - Stefan-Boltzmann's law
    - Linear approximation for small ΔT



- Power of heat transport
  - Heat transmission coefficient α
    - Typical value $6\ \mathrm{W\ m^{-2}\ K^{-1}}$
  - Scaling
    - Surface area A
    - Temperature difference ΔT

$$P = \alpha A \Delta T$$

3.5.4. Heat Transfer

- Interfaces
  - Solid
  - Fluid
- T-gradient: ΔT

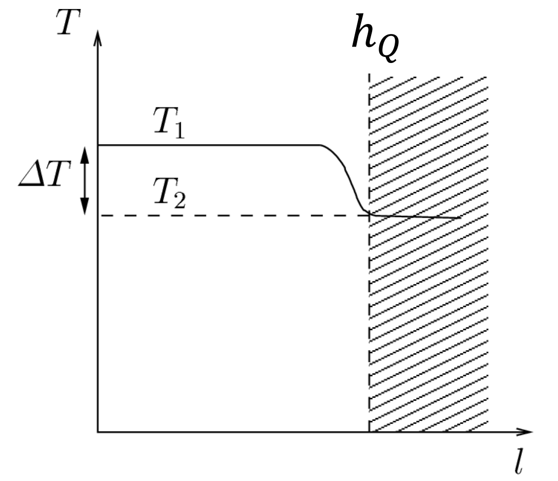


Fig. 3.32. Typical temperature curve for heat transmission between temperatures T_1 and T_2

- Flow density of heat transfer

$$j_Q = h_Q \Delta T$$

- Heat transfer coefficient h_Q

3.5.4. Heat Transition

- Heat transition
  - Two fluid reservoirs
  - Intermediate wall
  - Power

$$P = k_Q A \Delta T$$

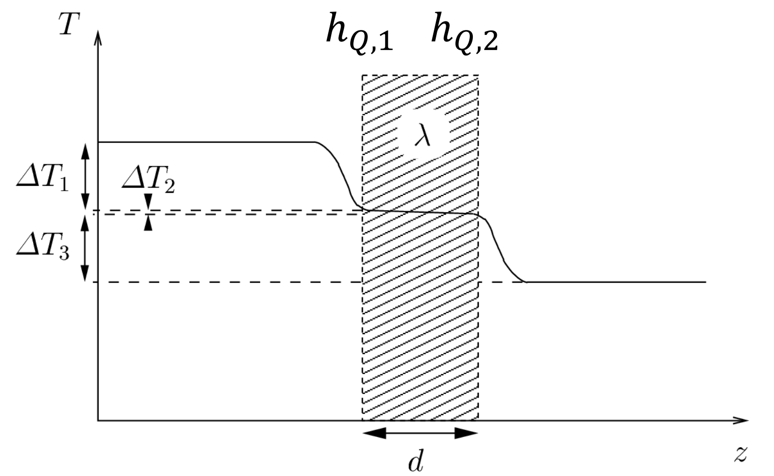


Fig. 3.33. Transition of heat between two vessels at T_1 and T_3 separated by a wall

$$\frac{1}{k_Q} = \frac{1}{h_{Q,1}} + \frac{d}{\lambda} + \frac{1}{h_{Q,2}}$$

- **Heat transition coefficient**
  - „Parallel circuit“ style equation
  - **Thermal conductivity λ**
  - **Heat transmission coefficient $h_{Q,i}$**
  - **Thickness of wall d**
    - **Efficient transition in microsystems**

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3.5.5. Newton‘s Law of Cooling

- Thermally undisturbed system
- Non-uniform temperature distribution T
- Thermal relaxation
- Newton‘s law of cooling
  - Time scale for thermal relaxation
  - Exponential decay

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{h_Q A}{C_m m} \Delta T$$

$$\Delta T = \Delta T_0 \mathrm{e}^{-t/\tau}$$

- **Characteristic decay time**
  - **Mass** (~ volume)
  - **Surface area**
  - **Fast relaxation in µfluidics**

$$\tau = \frac{C_m m}{h_Q A}$$

3.5. Heat Transport

optional

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3.5.6. Dimensionless Parameters

optional

$$Sc = \frac{\eta}{\varrho D} = \frac{v}{D} \overset{\text{id. gas}}{=} 1$$

- Peclet number

$$Pe_D = ReSc = \frac{vl}{D} = 4\left(\frac{l}{l_D}\right)^2$$

  - Convective particle transport
    vs
  - Diffusive particle transport

$$4\left(\frac{l}{l_D}\right)^2 = \frac{4l^2}{4Dt} = \frac{l^2}{Dl/v} = Pe$$

  - For heat diffusion

$$Pe_T = RePr = \frac{vl}{\lambda_D}$$

3.5.6. Dimensionless Parameters

optional

- Grashof number
  - Thermal buoyancy vs
  - Viscous forces

$$Gr = \frac{|\boldsymbol{g}|\alpha_V(T_1 - T_2)l^3}{\upsilon^2}$$

- Richardson number
  - Buoyancy vs
  - Inertia
  - Transition
    - Free buoyancy-driven flow ($Ri \ll 1$)
    - Forced convection flow ($Ri \gg 1$)

$$Ri = \frac{g\alpha_V \Delta T l}{\upsilon^2}$$

- Rayleigh number
  - Thermal buoyancy vs
  - Heat diffusion

$$Ra = PrGr$$

$$Pr = \frac{C_m \eta}{\lambda} = \frac{\upsilon}{\lambda_D}$$

3.5.6. Nusselt Number

optional

- Convective mass transport
  - E.g., turbulences
  - Accelerated heat transport
- Mere laminar flow
  - Transversal **temperature profile**
  - Large *T*-gradient
    - **Turbulences / convection** induced in boundary layer
- Nusselt number
  - Heat transfer in boundary layer
    - Convection
    - Diffusion
  - Heat transmission
    - From wall to fluid
    - Characteristic length *l*
    - Theoretical film thickness $\delta_Q = \lambda / h_Q$

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \lambda_D \Delta T + Q'''$$

energy / temperature equation

$$Nu = \frac{Q_{\mathrm{conv}}}{Q_{\mathrm{diff}}}$$

$$Nu = \frac{h_Q l}{\lambda} = \frac{l}{\delta_Q}$$

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3.5.7. Forced-Convection Heat Transfer

optional

$$Nu = \frac{Q_{\text{conv}}}{Q_{\text{diff}}}$$

- Functional relationship
  - Nusselt number Nu
  - Reynolds number Re
  - Prandtl number Pr

$$Re = \frac{\varrho_\infty \tilde{v} \tilde{l}}{\eta} = \frac{\tilde{v} \tilde{l}}{\upsilon}$$

$$Pr = \frac{C_m \eta}{\lambda} = \frac{\upsilon}{\lambda_D}$$

- Heat transfer characteristics
  - Boundary conditions
    - Shape
    - Surface roughness
      - Enhanced heat transfer
  - Boundary layer
    - Flow regime
    - Temperature profile
    - Velocity profile

3.5.7. Forced-Convection Heat Transfer

optional

- Thin films
  - Model for critical boundary layer
  - Decisive for heat transfer from wall
  - $Nu = Nu(Re, Pr)$
  - Turbulent regime
- Fluid film spreading on solid at rest
  - Shear stress τ
    - Friction
  - Friction coefficient
  - Relative layer thickness

$$\tau = \eta \frac{\partial v_z}{\partial y} \simeq \eta \frac{v}{\delta}$$

$$f_\epsilon = f_\epsilon(Re, \frac{\epsilon}{d}, A) = \frac{\tau_{\text{wall}}}{\frac{1}{2}\varrho v^2}$$

$$\frac{\delta}{d} = \frac{2\eta}{f_\epsilon \varrho v d} = \frac{2}{f_\epsilon Re}$$

3.5.7. Turbulent Flow of Thin Films

optional

$$\frac{\delta}{d} = \frac{2\eta}{f_\epsilon \varrho v d} = \frac{2}{f_\epsilon Re}$$

- Rewriting *Nu*

$$Nu = \frac{h_Q d}{\lambda} = \frac{d}{\delta}\frac{\delta}{\delta_Q} = \frac{1}{2} f_\epsilon Re \frac{\delta}{\delta_Q}$$

- Using (derivation not shown) $Pr^{1/3} = \delta / \delta_Q$

$$Nu = \frac{1}{2} f_\epsilon Re Pr^{1/3}$$

3.5.7. Turbulent Flow of Thin Films

optional

$$Nu = \frac{1}{2} f_\epsilon Re Pr^{1/3}$$

- Chilton-Colburn analogy

$$j_\mathrm{H} = \frac{f_\epsilon}{2}$$

$$j_\mathrm{H} = \frac{Nu}{Re Pr^{1/3}}$$

Dimensionless *j* - factor

- From thin film theory

$$Nu \simeq 0.04 Re^{3/4} Pr^{1/3}$$

$$Re = \frac{\varrho_\infty \tilde{v} \tilde{l}}{\eta} = \frac{\tilde{v} \tilde{l}}{\upsilon}$$

$$Pr = \frac{C_m \eta}{\lambda} = \frac{\upsilon}{\lambda_D}$$

3.5.7. Laminar Flow in a Tube

optional

- Fully developed flow
  - *Nu* function of axial coordinate *z*
  - Averaged value

- Thermal boundary conditions
  - Uniform wall temperature

$$Nu(z) = 1.077 Gz(z)^{1/3} \text{ for } Gz > 100$$
$$Nu = 1.61 Gz^{1/3} \text{ for } Gz > 1000$$

  - Uniform heat flux

$$Nu(z) = 1.302 Gz(z)^{1/3} \text{ for } Gz > 10^4$$
$$Nu = 1.953 Gz^{1/3} \text{ for } Gz > 10^4$$

3.5.7. Laminar Flow in a Tube

optional

- Sieder-Tate relationship
  - Simultaneously developing profiles
    - Temperature
    - Velocity
    - Viscosity on wall region η_{wall}

$$Nu = 1.86 Gz^{1/3} \left(\frac{\eta}{\eta_{\mathrm{wall}}} \right)^{0.14} \quad \text{for } Gz \left(\frac{\eta}{\eta_{\mathrm{wall}}} \right)^{0.14} \geq 2$$

- Fully developed profiles
  - Constant wall temperature

$$Nu = 3.66$$

  - Constant heat flux

$$Nu = 4.36$$

3.5.7. Turbulent Flow through a Tube

optional

- Fully developed flow
- Re: 10,000

- Nusselt number
  - Colburn equation

$$Nu = 0.023 Re^{0.8} Pr^{0.4}$$

  - Dittus-Boelter relation
    - More refined

$$Nu = 0.0225 Re^{0.795} Pr^{0.495} \mathrm{e}^{-0.0225(\ln Pr)^2}$$

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3.5.8. Heat Transfer in Microchannels

optional

- Convective cooling by microchannels
  - Important technical topic over last decades
    - Electronic devices
    - Refrigerators

- Microchannels
  - Deviation from straight-forward macro-theory
  - Reduced critical Reynolds number
    - Premature transition to turbulent regime
  - Example: artificially roughened surfaces
    - Turbulent mixing in boundary layers
    - Improved heat transfer characteristics

3.5.8. Heat Transfer in Microchannels

optional

- Fully developed laminar flow
  - Empirical formula

$$Nu = 9.72 \times 10^{-4} Re^{1.17} Pr^{1/3}$$

Compare macro:

$$Nu(z) = 1.077 Gz(z)^{1/3} \text{ for } Gz > 100$$
$$Nu = 1.61 Gz^{1/3} \text{ for } Gz > 1000$$

- Turbulent regime

$$Nu = 3.82 \times 10^{-6} Re^{1.96} Pr^{1/3}$$
$$Nu = 0.00805 Re^{4/5} Pr^{1/3}$$

Compare macro:

$$Nu = 0.023 Re^{0.8} Pr^{0.4}$$

3.5.8. Heat Transfer in Microchannels

optional

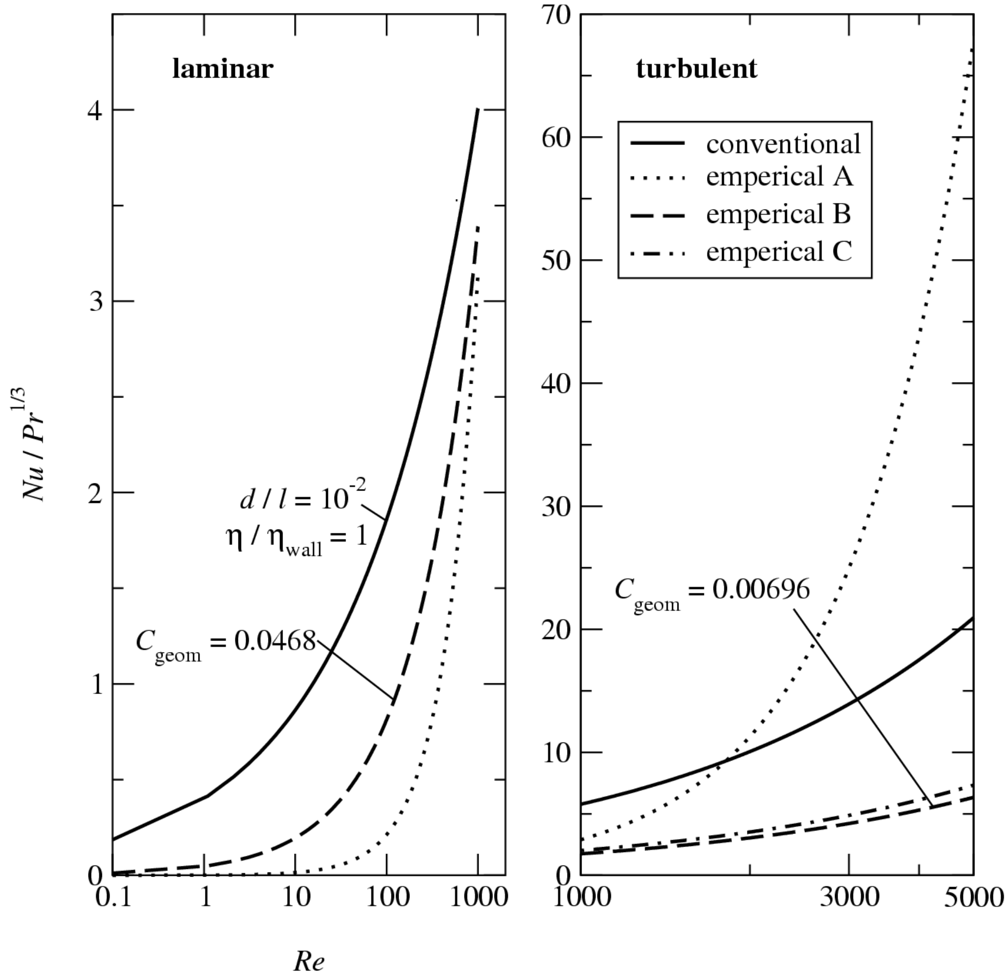


Fig. 3.34. Recalibrated Nusselt number $Nu/Pr^{1/3}$ as a function of the Reynolds number Re in the laminar (left) and turbulent (right) flow regime. The empirical formulas A, B and C have been adopted from , (with C_{geom} calculated for an aspect ratio of 1 and $d_{\mathrm{hd}} = 200\,\mu\mathrm{m}$) and , respectively

Summary

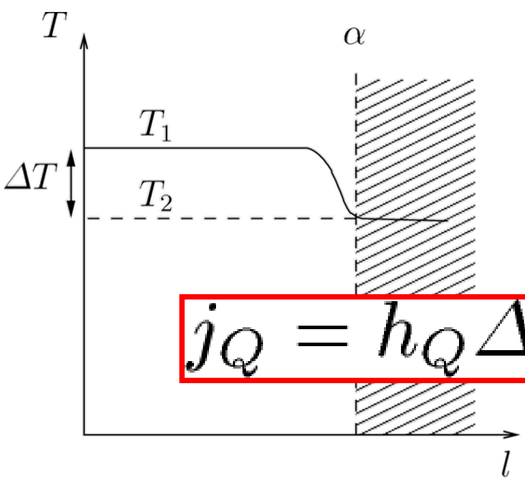
Coupling of T-distribution and flow field v with heat source Q'''

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \lambda_D \Delta T + Q'''$$

Boussinesq (only density ↦ buoyancy in Navier-Stokes

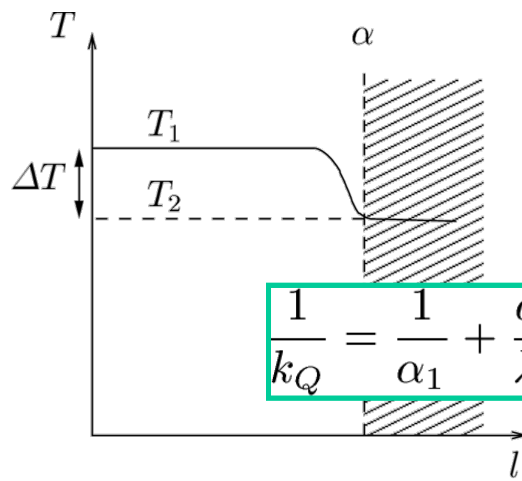
$$\varrho(T) = \varrho_0(1 - \alpha_V[T(\boldsymbol{x}, t) - T_0])$$

Heat transfer



$$j_Q = h_Q \Delta T$$

Heat transition



$$P = k_Q A \Delta T$$

$$\frac{1}{k_Q} = \frac{1}{\alpha_1} + \frac{d}{\lambda} + \frac{1}{\alpha_2}$$