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5. Fluidic Capacitance

3.4. Fluidic Networks

1. **Analogy to Electric Circuits**
2. Example: Simple Electric Circuit
3. Flow Resistance
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3.4.1. Analogy to Electric Circuits

- „Exact“ approach
  - Continuity of mass
  - Navier-Stokes equations
  - Equation of state
  - Boundary conditions

$$\frac{\partial}{\partial t}\varrho + \nabla \cdot (\varrho \boldsymbol{v}) = 0$$

$$\varrho_\infty \left[\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p + \eta \nabla^2 \boldsymbol{v} + \varrho_\infty \boldsymbol{g}$$

$$pV = nR_\mathrm{g}T$$

$$v_{||} = 0 \quad v_\perp = 0$$

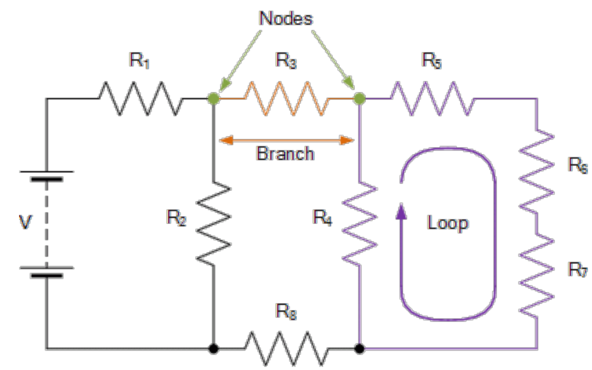
- Analytical Solution
  - Only for cases of high symmetry

- **Numerical Solution**
  - Discretization
    - Tremendous number of lattice points
  - High computational effort, e.g.,
    - Spatio-temporally varying lattice
    - Convergence problems
  - **Most problems require further simplification!**

3.4.1. Analogy to Electric Circuits

- Physics of electric circuits
  - Conduction band
  - Some 10^{23} electrons
  - QM ⇒ Hamiltonian
  - Boundary conditions

HOPELESS!



- Electrical engineering ⇒ Parameter reduction
  - Elements condensed to discrete **points**
  - One-dimensional current I
  - Voltage U: Potential difference between points
  - **Discrete** set of idealized elements
    - Resistance R
    - Inductance L
    - Capacitance C

Kirchhoff's laws

$$\sum_i I_i = 0 \text{ at each node}$$

$$\sum_i U_i = 0 \text{ along any closed mesh}$$

3.4.1. Fluidic Analogs

- Kirchhoff's conservation laws: Quantities
- Current $I \mapsto I_m$
  - Mass flow with density ρ
  - Flow velocity v

$$I_m = \int_A \varrho \mathbf{v} \, \mathrm{d}\mathbf{A}$$

- Total energy (density)
  - Potential (energy) $U \mapsto p$
  - Kinetic energy $E_{\mathrm{kin}} \mapsto \frac{1}{2}\rho v^2$
- Hydrodynamics
  - No energy dissipation
  - Recall: Bernoulli

$$p + \frac{\rho}{2} v^2 = p_{\mathrm{tot}} = \mathrm{const.}$$

- Hydrodynamic potential $U_{\mathrm{hd}} \mapsto p_{\mathrm{tot}}$
  - Preserved in flows

$$U_{\mathrm{hd}} = \frac{p}{\rho} + \frac{1}{2} v^2 = \frac{p_{\mathrm{tot}}}{\rho} = \mathrm{const.}$$

3.4.1. Fluidic Potential

- Rewrite potential
  - Total pressure p_{tot} at heart of U_{hd}

$$U_{\text{hd}} = \frac{p}{\varrho} + \frac{\bar{v}^2}{2} = \frac{p_{\text{tot}}}{\varrho} \longrightarrow$$

Bernoulli

$$p + \frac{\varrho}{2} v^2 = p_{\text{tot}} = \text{const.}$$

$$U_{\text{hd}} = \frac{1}{\varrho}(p + p_A)$$

$$p_A = \frac{\varrho \bar{v}^2}{2} = \frac{I_m^2}{2 \varrho A^2}$$

$$\bar{v} = I_m / A$$

  - p_A : pressure drop at element of cross-section A

- Differential definition

$$\mathrm{d}U_{\text{hd}} = \frac{1}{\varrho}(\mathrm{d}p + \mathrm{d}p_A) = \frac{1}{\varrho}\left(\mathrm{d}p + \frac{\partial p_A}{\partial I_m}\mathrm{d}I_m\right)$$

$$\mathrm{d}U_{\text{hd}} = \frac{1}{\varrho}(\mathrm{d}p + R_A \mathrm{d}I_m)$$

$$R_A \stackrel{\text{def}}{=} \frac{\partial p_A}{\partial I_m} = \frac{I_m}{\varrho A^2}$$

- **$\mathrm{d}\rho^{-1} p$ approximates $\mathrm{d}U_{\mathrm{hd}}$ for small variations $R_A I_m$!**

3.4.1. Analogy to Electric Circuits

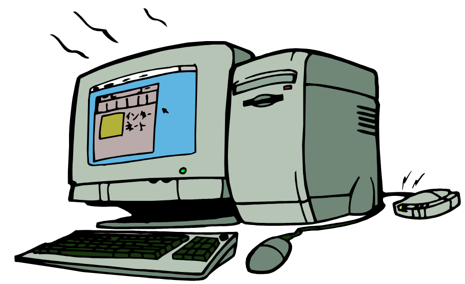
- Hydrodynamic power
  - Units of Watt
  - $1\ \mathrm{W} = 1\ \mathrm{N\ m\ s^{-1}}$

$$P_{\mathrm{hd}} = I_m U_{\mathrm{hd}} = \frac{I_m}{\varrho}(p + R_A I_m) = \frac{I_m}{\varrho} p_{\mathrm{tot}}$$

- Transfer functions
  - Workhorse of electric circuits
  - Relation between
    - Potential drop U_{hd}
    - Current I_m
  - Complex number $A \cdot e^{i\phi}$
    - Amplitude A
    - Phase shift ϕ
  - Derivation
    - Analytical: for rather abstract components (e.g., Ohm‘s laws)
      - Resistors
      - Capacitances
      - Inductivities
    - Numerical: (CFD)

3.4.1. Analogy to Electric Circuits

- Net list
  - Alignment of components
  - Internal coupling



- Differential equation for entire system
  - Net list
  - Transfer functions
  - Conservation laws

- Computation of network
  - Numerical procedures (SPICE, SABER)
  - Solution at discrete locations

3.4. Fluidic Networks

1. Analogy to Electric Circuits
2. **Example: Simple Electric Circuit**
3. Flow Resistance
4. Fluidic Inertance
5. Fluidic Capacitance

3.4.2. Example: Simple Electric Circuit

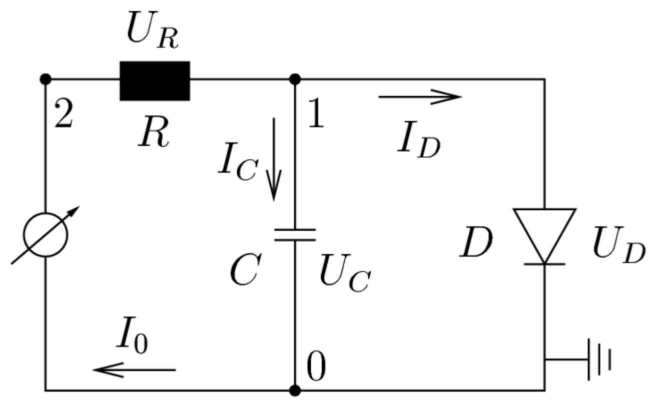


Fig. 3.25. Equivalent network for an electric circuit constituted by a resistor R, a capacitor C and a diode D

<u>Transfer functions</u>

$$I_R = \frac{U_R}{R}$$

$$I_C = C\frac{\mathrm{d}U_C}{\mathrm{d}t}$$

$$I_D = I_{\mathrm{sat}}\left(\mathrm{e}^{\frac{U_D}{U_T}} - 1\right)$$

<u>Net list</u>

$0 \mapsto 2$: current $I_0 = I_R$

$1 \mapsto 2$: resistance R

$0 \mapsto 1$: capacitance C

$0 \mapsto 1$: diode current I_D

3.4.2. Example: Simple Electric Circuit

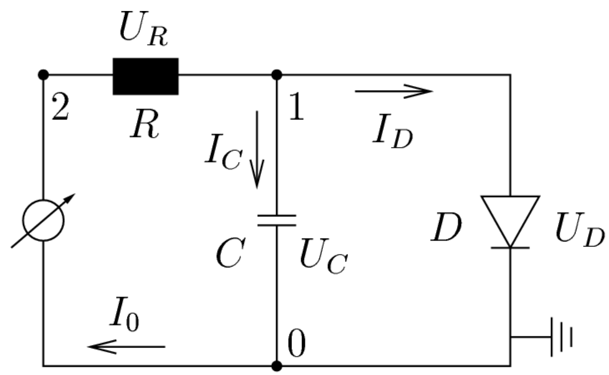


Fig. 3.25. Equivalent network for an electric circuit constituting by a resistor R, a capacitor C and a diode D

- Kirchhoff's mesh rule

$$U_C = U_D$$

- Kirchhoff's node rule

$$\text{node } 1: -\frac{U_R}{R} + C\frac{\mathrm{d}U_C}{\mathrm{d}t} + I_{\mathrm{sat}}\left(\mathrm{e}^{\frac{U_D}{U_T}} - 1\right) = 0$$

$$\text{node } 2: \frac{U_R}{R} - I_0 = 0$$

- DE for system

$$C\frac{\mathrm{d}U_C}{\mathrm{d}t} + I_{\mathrm{sat}}\left(\mathrm{e}^{\frac{U_C}{U_T}} - 1\right) = I_0$$

3.4. Fluidic Networks

1. Analogy to Electric Circuits
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3.4.3. Flow Resistance

- Fluidic equivalent to ideal Ohmic resistor
  - Quotient
    - „Potential" difference
    - „Current"
  - Referring to mass flow I_m
  - Alternative to volume flow I_V
    - Attention: Compressibility!

$$R_{\mathrm{hd}} = \frac{\Delta p}{I_m}$$

- Mechanism
  - Viscosity
  - Boundary conditions
  - Character of flow

- Note
  - **No energy dissipation (heat)** like Ohmic resistor
  - Conversion between
    - Potential energy ($\propto$ pressure $U_{\mathrm{hd}} \mapsto p$)
    - Kinetic energy ($\propto$ square of mass flow I_m)

3.4.3. Flow Resistance

- Analytical solution for tube

$$I_V = \frac{\pi}{8\eta}\frac{\Delta p}{l} r_0^4$$

Hagen-Poiseuille

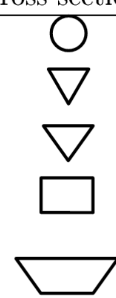
$$R = \frac{U}{I}$$

Ohm

- Hydrodynamic resistance

$$R_{\mathrm{hd}} = C_{\mathrm{nc}} \frac{\eta l}{\varrho A^2}$$

  - Numerical coefficient
    - Shape of cross-section



cross-section	class	C_{nc}
	circular law of Hagen-Poiseuille	$8\pi = 25.13$
	equilateral triangle	$\frac{60}{\sqrt{3}} = 34.64$
	KOH-etched triangle < 100 >	35.12
	rectangle	$\frac{2}{\mathcal{A}\sum_{i=1}^{\infty}\frac{\mathcal{A}}{\alpha_i^5}\left(\frac{\alpha_i}{\mathcal{A}}-\tanh\frac{\alpha_i}{\mathcal{A}}\right)}$
	KOH-etched trapezoid	$\frac{12-1.38\mathcal{A}+4\mathcal{A}^2}{\mathcal{A}-0.85\mathcal{A}^2+0.28\mathcal{A}^3}$

Table 3.4. Correction coefficients C_{nc} for flow resistance of non-circular channel cross-sections as derived in the laminar regime, adopted from . The parameter $\mathcal{A}$ (aspect ratio for rectangles) and α_i for the rectangular cross section are defined as $\mathcal{A}$ = height/(average width) and $\alpha_i = \pi(2i-1)/2$. A good approximation for the rectangular cross-section is given by $C_{\mathrm{nc}} = 8(1+\mathcal{A})^2/\mathcal{A}$

3.4.3. Flow Resistance

optional

- Inlet resistance
  - Distance to reach fully developed flow profile z_{devel}
  - Inertia
    - Acceleration from rest to asymptotic velocity
  - „Direct current“ (DC) phenomena

$$R_{\mathrm{hd}} = R_{\mathrm{hd},0} + \boxed{K \frac{I_m}{2\varrho A^2}}$$

Nonlinear term

3.4.3. Orifice Plate

optional

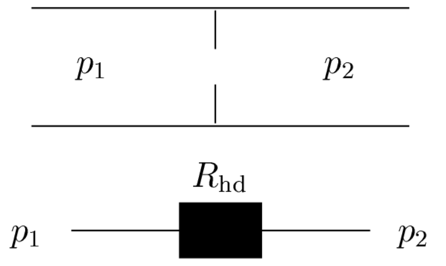


Fig. 3.26. Geometrically, an orifice plate is a fluidic constraint of negligible length (top). In a fluidic network it represents a hydraulic resistance R_{hd} (bottom)

- Fluidic constraint
  - „Direct current“ phenomenon
  - Negligible length
  - Vanishing mass resides in aperture
    - No fluidic inertance
- Idealized conditions
  - Perfect laminar flow
  - Perfect conversion: potential ↦ kinetic ↦ potential energy
- Real system
  - Energy losses during reconversion
  - Coefficient C_{d} depends on various parameters (also Δp)

$$R_{\mathrm{hd}} = \frac{1}{C_{\mathrm{d}} A} \sqrt{\frac{\Delta p}{2\varrho}}$$

3.4.3. Nozzles

optional

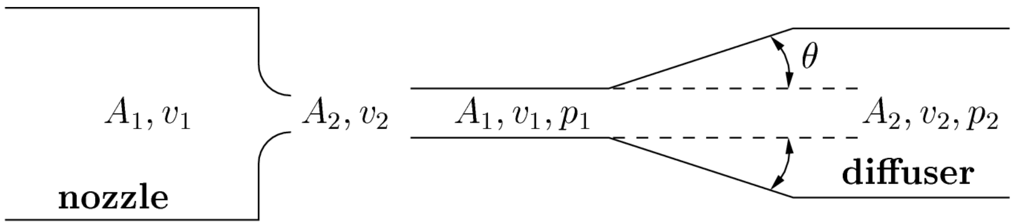


Fig. 3.27. Schematic of a nozzle (left) and a diffuser (right). The flow goes from left to right with an initial cross-section A_1 and velocity v_1 changing to A_2 and v_2 after leaving the element

- Geometrical contraction
  - Cross-section A_1 and A_2
- Idle ($A_2 = 0$)
  - R_{hd} infinite
- Case $A_1 = A_2$
  - R_{hd} vanishes
- Scaling with I_m

$$R_{hd} = \frac{I_m}{2\varrho A_2^2}\left[1 - \left(\frac{A_2}{A_1}\right)^2\right]$$

3.4. Fluidic Networks

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3.4.4. Fluidic Inertance

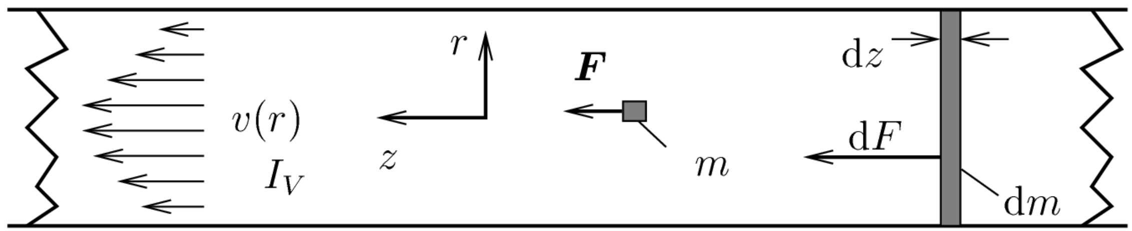

Fig. 3.28. Inertance derived from the force $\mathrm{d}F$ acting on an active element of mass $\mathrm{d}m$ and longitudinal width $\mathrm{d}z$

$F=ma$

$$dF = A\mathrm{d}p = \int_{r=0}^{r=r_0} 2\pi\varrho\frac{\mathrm{d}v(r)}{\mathrm{d}t} r\mathrm{d}r\mathrm{d}z = \mathrm{d}z\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left(2\pi\varrho\int_{r=0}^{r=r_0} v(r)r\mathrm{d}r\right)}_{I_m}$$

$$dF = \mathrm{d}z\frac{\mathrm{d}I_m}{\mathrm{d}t}$$

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{1}{A}\frac{\mathrm{d}I_m}{\mathrm{d}t}$$

$$\Delta p_m = \frac{l}{A}\frac{\mathrm{d}I_m}{\mathrm{d}t} \stackrel{\mathrm{def}}{=} L_{\mathrm{hd}}\frac{\mathrm{d}I_m}{\mathrm{d}t}$$

$$L_{\mathrm{hd}} = \frac{l}{A}$$

- Setting fluidic mass into motion
- AC phenomenon
- Conversion
  - Pressure
    - Potential energy
  - Fluid flow
    - Kinetic energy
- **How tightly can mass flow follow pressure signal?**

3.4.3. Fluidic Inertance

- Reaction of fluid
  - Speed of sound: $1000\ \mathrm{m\ s^{-1}}$
  - Channel length: $10\ \mathrm{mm}$
  - Time for pressure signal: $10\ \mu\mathrm{s}$

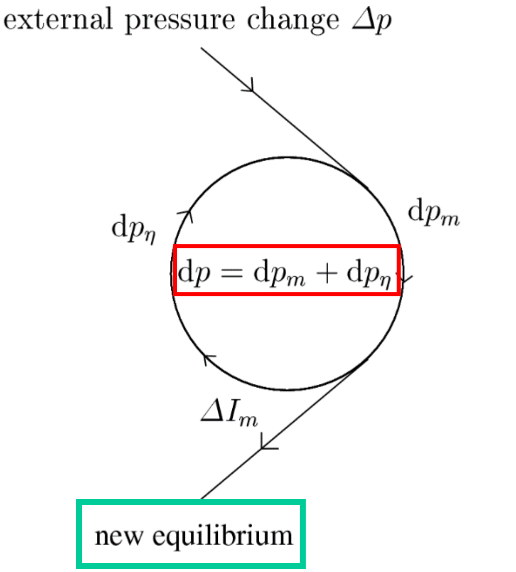

Fig. 3.29. Response of the mass flow I_m of a fluidic system to an external pressure change $\Delta p = \Delta p_m + \Delta p_\eta$

- Onset of flow
  - Pressure drops
    - Viscosity p_η
    - Inertia p_m
  - Dynamics of system

$$\mathrm{d}p = \mathrm{d}p_\eta + \mathrm{d}p_m = R_{\mathrm{hd}}\mathrm{d}I_m + L_{\mathrm{hd}}\mathrm{d}I_m$$

3.4.3. Fluidic Inertance

- Reaction of fluid
  - Speed of sound: 1000 m s^{-1}
  - Channel length: 10 mm
  - Time for pressure signal: 10 µs

- Onset of flow
  - Pressure drops
    - Viscosity p_η
    - Inertia p_m
  - Dynamics of system

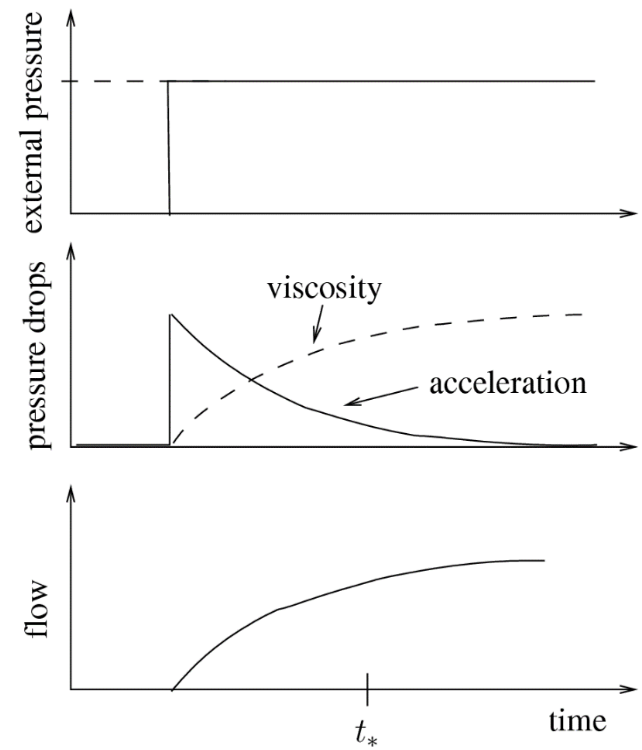

Fig. 3.30. The external pressure applied to a fluidic system split into contributions to overcome viscosity $\mathrm{d}p_\eta$ and acceleration $\mathrm{d}p_m$

$$\mathrm{d}p = \mathrm{d}p_\eta + \mathrm{d}p_m = R_{\mathrm{hd}}\mathrm{d}I_m + L_{\mathrm{hd}}\mathrm{d}I_m$$

3.4.3. Fluidic Inertance

- Characteristic response time

$$\tilde{t} = \frac{L_{\mathrm{hd}}}{R_{\mathrm{hd}}}$$

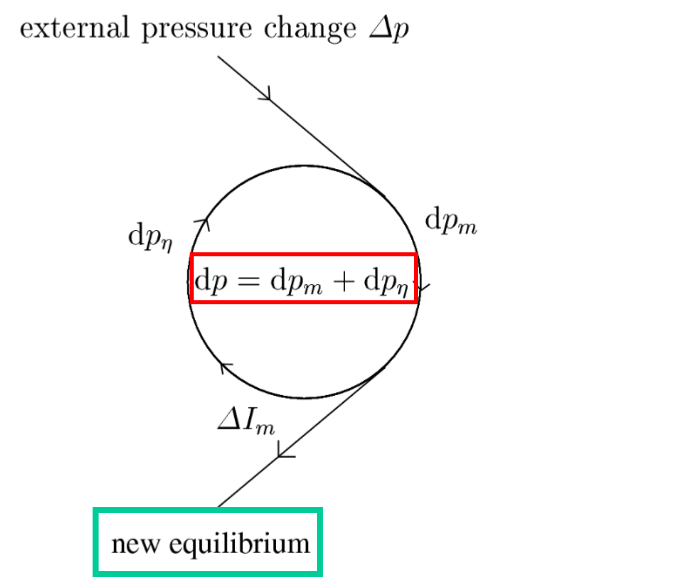

Fig. 3.29. Response of the mass flow I_m of a fluidic system to an external pressure change $\Delta p = \Delta p_m + \Delta p_\eta$

- Rewritten $$\tilde{t} = \frac{A}{C_{\mathrm{nc}}\upsilon} = \frac{\pi r_0^2}{C_{\mathrm{nc}}\upsilon}$$
  - Independent of channel length
  - Scaling with r_0^2

- Example with typical values
  - $\upsilon = 10^{-6}\ \mathrm{m^2\,s^{-1}}$
  - $r_0 = 400\ \mu\mathrm{m}$
  - Response time of **20 ms ≈ 10^{-2} s**
  - **Typical frequencies in μfluidic circuits**
    - ~**100 Hz**

3.4. Fluidic Networks

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5. **Fluidic Capacitance**

3.4.5. Fluidic Capacitance

- **Elastic** components
  - Membranes
  - Compressible fluids
  - AC & DC phenomenon
- External pressure
  - Intermediate „storage" of mass

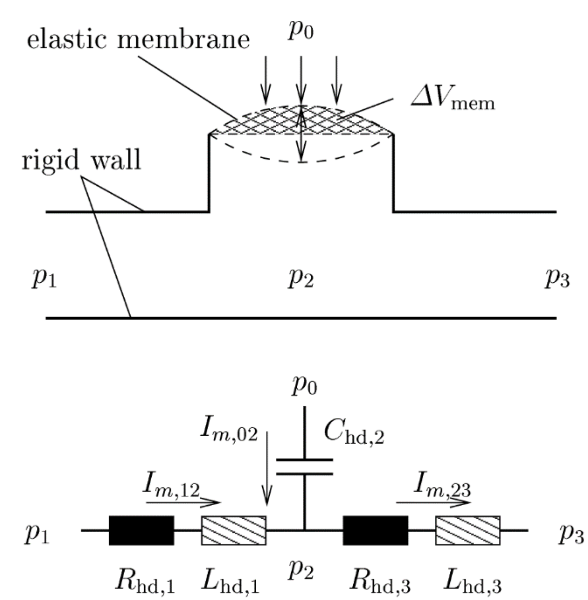

Fig. 3.31. Elastic membrane acting as fluidic capacitance. (top) The membrane volume ΔV_{mem} adjusts to the difference Δp between the external and internal pressure p_0 and p_2, respectively. (bottom) Electric circuit equivalent of the elastic element with hydraulic resistances $R_{\mathrm{hd},i}$, inertances $L_{\mathrm{hd},i}$ and capacitance $C_{\mathrm{hd},i}$ governing flows $I_{m,ij}$ in the corresponding channel sections

$$\Delta m = \Delta(\varrho V) \stackrel{\mathrm{def}}{=} C_{\mathrm{hd}} \Delta p$$

flow rate

$$I_m = \frac{\mathrm{d}m}{\mathrm{d}t} = V \frac{\mathrm{d}\varrho}{\mathrm{d}t} + \varrho \frac{\mathrm{d}V}{\mathrm{d}t} = C_{\mathrm{hd}} \frac{\mathrm{d}p}{\mathrm{d}t}$$

3.4.5. Compressible Fluids

- Finite compressibility κ
  - **Flow** by compression of fluid of density ϱ

$$\boxed{I_m} = V_0 \frac{\mathrm{d}\varrho}{\mathrm{d}t} = V_0 \frac{\partial \varrho}{\partial p} \frac{\mathrm{d}p}{\mathrm{d}t} = \varrho_0 V_0 \boxed{\kappa} \frac{\mathrm{d}p}{\mathrm{d}t}$$

$$m_0 = \varrho_0 \cdot V_0$$

- Hydraulic capacitance

$$\boxed{C_{\mathrm{hd}} = m_0 \boxed{\kappa}}$$

3.4.5. Elastic Membranes

- Unknown variables
  - Three currents $I_{m,i}$
  - Pressure p

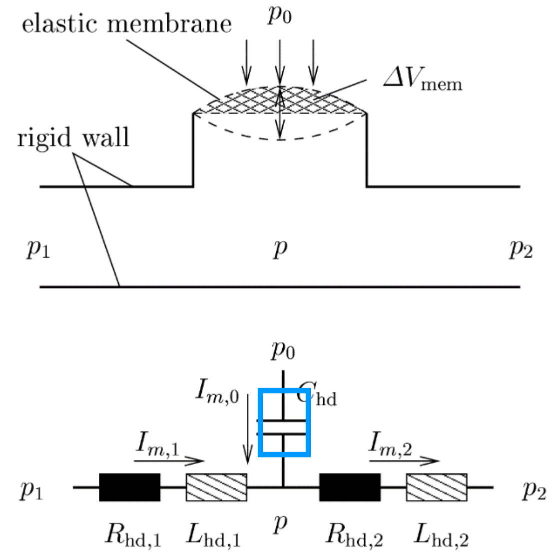

Fig. 3.31. Elastic membrane acting as fluidic capacitance. (top) The membrane volume ΔV_{mem} adjusts to the difference Δp between the external and internal pressure p_0 and p_2, respectively. (bottom) Electric circuit equivalent of the elastic element with hydraulic resistances $R_{\text{hd},i}$, inertances $L_{\text{hd},i}$ and capacitance $C_{\text{hd},i}$ governing flows $I_{m,ij}$ in the corresponding channel sections

- Kirchhoff's mesh rule

$$p_1(t) - p_2(t) = R_{\text{hd},1} I_{m,1}(t) + L_{\text{hd},1} \frac{\mathrm{d}I_{m,1}(t)}{\mathrm{d}t} + R_{\text{hd},2} I_{m,2}(t) + L_{\text{hd},2} \frac{\mathrm{d}I_{m,2}(t)}{\mathrm{d}t}$$

- Flow in the side channel

$$I_{m,0}(t) = \frac{\mathrm{d}V_{\text{mem}}}{\mathrm{d}t} = C_{\text{hd}} \left[\frac{\mathrm{d}p(t)}{\mathrm{d}t} - \frac{\mathrm{d}p_0(t)}{\mathrm{d}t} \right]$$

3.4.5. Elastic Membranes

- Membrane function
  - Flow by **elastic** expansion

$$\Delta V_{\text{mem}}(t) \simeq k_{\text{elast}}[p(t) - p_0(t)]$$
$$\text{with } C_{\text{hd}} = \varrho k_{\text{elast}}$$

- Current in side channel

$$I_{m,0}(t) + I_{m,1}(t) = I_{m,2}(t)$$

- DE for system with one unknown
  - Combination of four equations

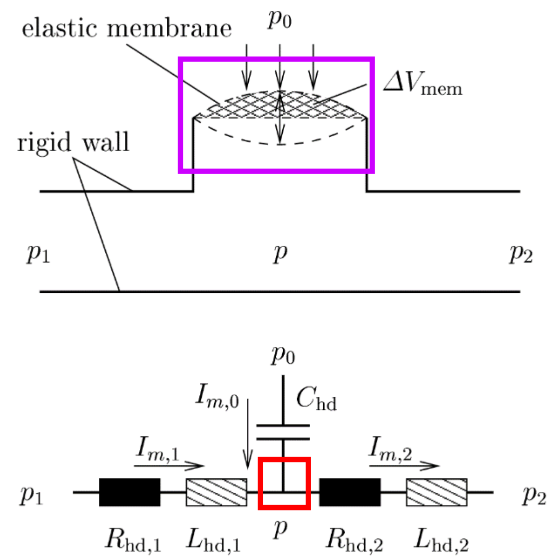

Fig. 3.31. Elastic membrane acting as fluidic capacitance. (top) The membrane volume ΔV_{mem} adjusts to the difference Δp between the external and internal pressure p_0 and p_2, respectively. (bottom) Electric circuit equivalent of the elastic element with hydraulic resistances $R_{\text{hd},i}$, inertances $L_{\text{hd},i}$ and capacitance $C_{\text{hd},i}$ governing flows $I_{m,ij}$ in the corresponding channel sections

Summary

Mass flow

$$I_m = \int_A \varrho \boldsymbol{v} \mathrm{d}\boldsymbol{A}$$

Voltage $U \leftrightarrow \Delta p$

$$U_{\mathrm{hd}} = \frac{p}{\rho} + \frac{1}{2} v^2 = \frac{p_{\mathrm{tot}}}{\rho} = \mathrm{const.}$$

Resistance

$$R_{\mathrm{hd}} = \frac{\Delta p}{I_m}$$

$$R_{\mathrm{hd}} = C_{\mathrm{nc}} \frac{\eta l}{\varrho A^2}$$

Inertance
(mass acceleration)

$$L_{\mathrm{hd}} = \frac{l}{A}$$

Capacitance

Compressibility

$$C_{\mathrm{hd}} = m_0 \kappa$$

Elastic membrane

$$C_{\mathrm{hd}} = \varrho k_{\mathrm{elast}}$$