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3. Physics of Microfluidic Systems

3.3. Fluid Dynamics

1. Dynamic Pressure
2. Cavitation
3. Coanda Effect
4. Hydrodynamic Forces
5. Pressure Waves
6. Flow through Constriction

3.3. Fluid Dynamics

1. **Dynamic Pressure**
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3.3.1. Bernoulli Equation

$$p + \frac{\varrho}{2} v^2 = p_{\mathrm{tot}} = \mathrm{const.}$$

Approx. in NS: Irrotational flow

- Non-stationary term $\frac{\partial v}{\partial t} = 0$
- Frictionless $\eta = 0$
- Discarding gravity $g = 0$

- Total pressure p_{tot}
  - Also stagnation pressure
  - Preserved in flow
- Static pressure p
  - Measured by hydrostatic pressure head
- Dynamic pressure $\propto v^2$

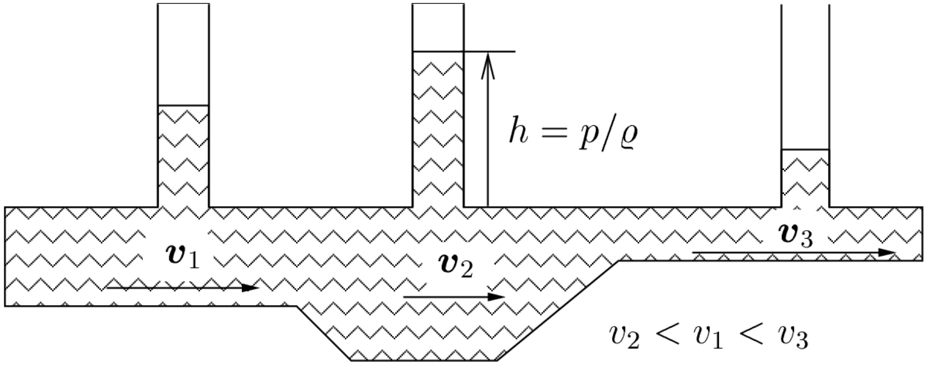

Fig. 3.20. Static pressure in a tube with varying diameter

3.3. Fluid Dynamics

1. Dynamic Pressure
2. **Cavitation**
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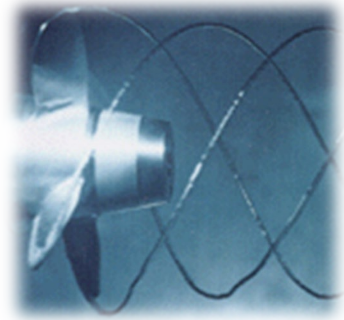
3.3.2. Cavitation

- Formation of bubbles at regions of high velocities
  - Related to Bernoulli equation
  - Vapor pressure p_{vap}
  - Static pressure $p < p_{\mathrm{vap}}$
  - Vaporization of liquid

$$p + \frac{\varrho}{2} v^2 = p_{\mathrm{tot}}$$



- Energy
  - Work against bending pressure
  - Stored energy $E = \sigma \cdot \Delta A$
  - Release of E
    - Local hot-spots
    - Chemical reactions
    - Corrosion
    - Emission of light
  - Possible detrimental to functionality of device

3.3. Fluid Dynamics

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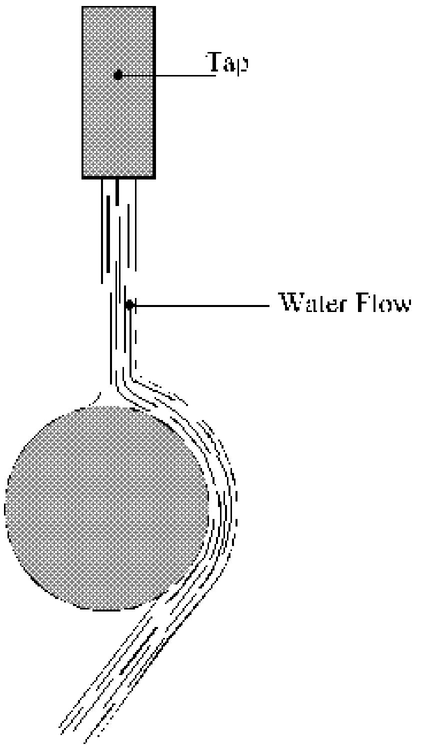
3.3.3. Coanda Effect

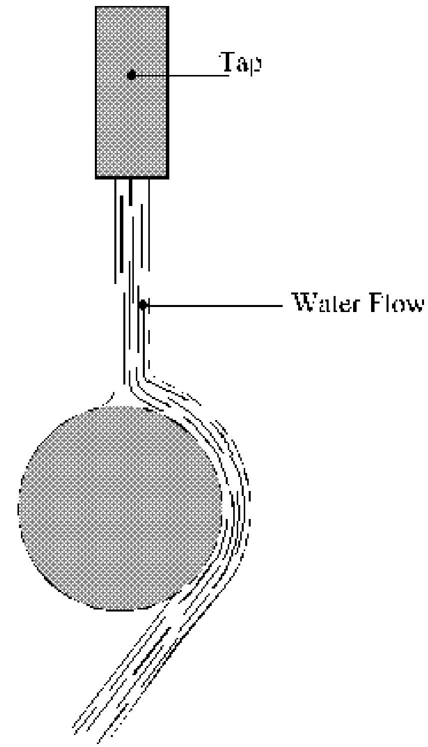
- Deflection of jets at curved surfaces
  - Discovered by Thomas Young in 1800
  - Rediscovered by Coanda in 1910
  - Understood in 1930
    - Dynamic pressure
  - In turbulent jets up to moderate Reynolds numbers
  - Curvature and angle not too sharp



$$p + \frac{\varrho}{2} v^2 = p_{\mathrm{tot}}$$

3.3.3. Coanda Effect

- Explanation
  - Velocities of turbulently moving particles far greater than jet speed
  - Underpressure
  - Nearby gas sucked into stream
  - Space between adjacent wall and jet evacuated
  - Jet tends to stick to wall



- Application
  - Flow switches
  - Fluidic amplifiers

$$p + \frac{\varrho}{2} v^2 = p_{\mathrm{tot}}$$

3.3. Fluid Dynamics

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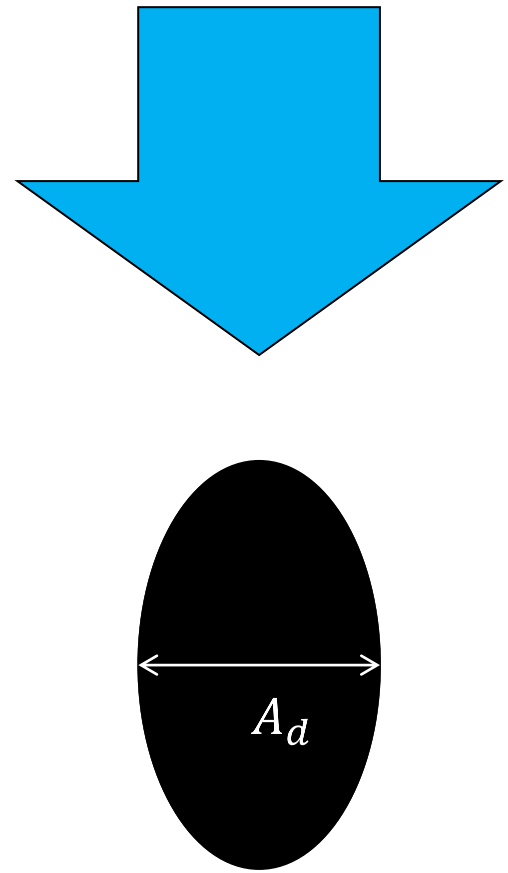
3.3.4. Drag Coefficient

$$F = \boxed{C_\mathrm{d}} \frac{\varrho}{2} v^2 A_\mathrm{d}$$

- Drag force F
  - Body in fluid stream at velocity v
  - Unconfined medium
  - Characteristic area (cross-section) A_d
  - Fluid density ϱ

- Drag coefficient C_d
  - Shape of body
  - Surface roughness
  - Reynolds number Re

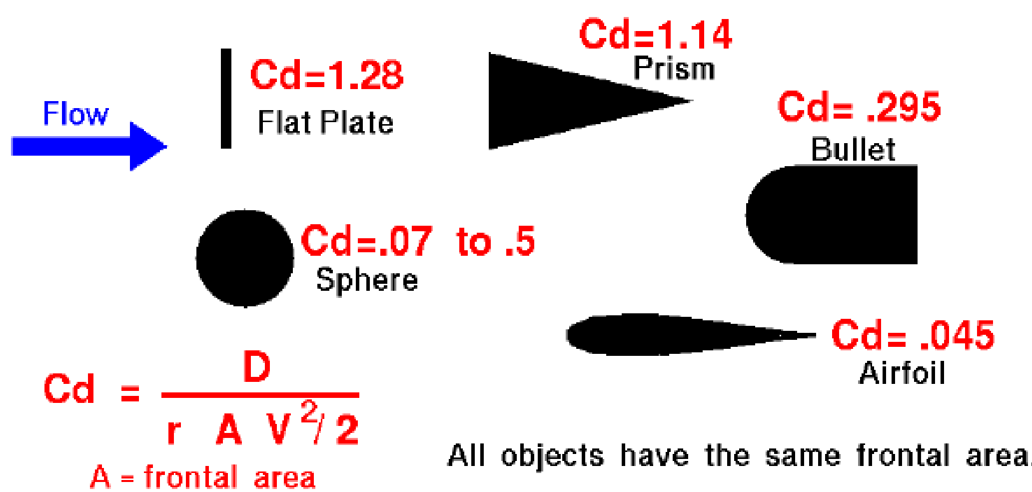


3.3.4. Drag Coefficient

$$F = \boxed{C_\mathrm{d}} \frac{\varrho}{2} v^2 A_\mathrm{d}$$

- Large velocity v
  - $C_\mathrm{d} \approx \mathrm{const.}$
  - $F \sim v^2$
- Low Re (laminar)
  - $C_\mathrm{d} \sim 1 / v$
  - $F \sim v$

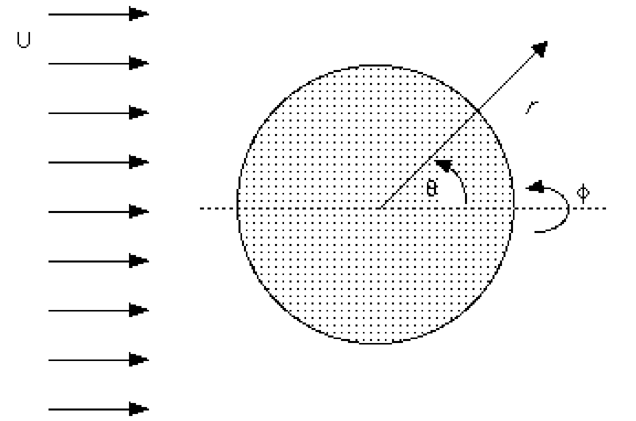
Examples:



3.3.4. Stokes Drag

$$F_{\mathrm{Stokes}} \simeq -\eta \frac{\mathrm{d}v}{\mathrm{d}z} 4\pi r_0^2 \simeq -4\pi\eta v r_0$$

- Force on sphere in fluid stream
  - Approximation by Stokes
  - Radius r_0
  - Relative speed v
  - Viscosity η
  - Laminar flow
  - Flow undisturbed at sufficient distance
  - $v = 0$ on surface of sphere



$$F_{\mathrm{Stokes}} = -6\pi\eta v r_0 = -\frac{6 A_{\mathrm{d}} \varrho v^2}{Re}$$

$$C_{\mathrm{d}} = \frac{12}{Re}$$

$$Re = \frac{\varrho_\infty \tilde{v} \tilde{l}}{\eta} = \frac{\tilde{v}\tilde{l}}{\upsilon}$$

more detailed calculation

3.3. Fluid Dynamics

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3.3.5. Pressure Waves

optional

- Interplay
  - Oscillating external pressure
  - Finite compressibility κ of fluid
  - Longitudinal modes only
    - No restoring force upon shear stress

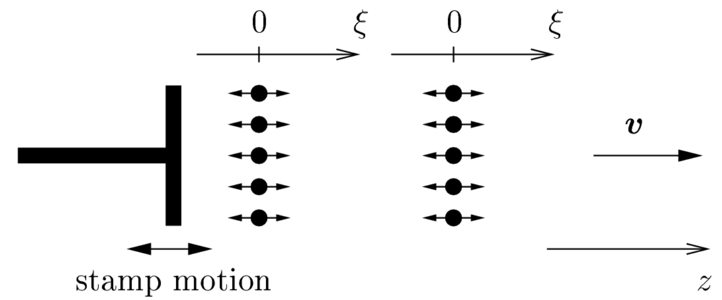

Fig. 3.23. Schematic of the sound wave generation by a tube. The stamp oscillates creating periodic density fluctuation which propagate with the velocity vector $\boldsymbol{v}$ in z-direction. The restoring force of the fluid results from its finite compressibility κ

- Density oscillations
  - Harmonic actuation
  - Angular frequency ω
  - Stamp amplitude ξ_0
  - Sound particle velocity v_ξ

$$\Delta\varrho = \frac{M_n k_z \xi_0}{\kappa R_g T} \cos(\omega t - k_z z)$$

- Phase velocity

$$c_\kappa = \lambda\nu = \frac{\omega}{k}$$

3.3.5. Wave Equation

optional

$$\triangle \xi = \frac{1}{c_\kappa^2}\frac{\partial^2 \xi}{\partial t^2}$$

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Laplacian

- Using

$$\frac{\Delta p}{\Delta z} \simeq -\frac{1}{\kappa}\frac{\partial^2 \xi}{\partial z^2} \qquad \frac{\Delta F}{\Delta z} = -A\frac{\Delta p}{\Delta z}$$

compare

- Relation

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\Delta F}{m} = -\frac{A \Delta z}{m}\frac{\Delta p}{\Delta z} = -\frac{V \Delta p}{m \Delta z} \simeq \frac{1}{\varrho \kappa}\frac{\partial^2 \xi}{\partial z^2}$$

Newton

$$c_\kappa = \sqrt{\frac{1}{\kappa \varrho}}$$

3.3.5. Pressure Waves

optional

- Typical values

$$c_\kappa = \sqrt{\frac{1}{\kappa \varrho}}$$

substance	temperature / °C	speed of sound / $\mathrm{m\,s^{-1}}$
hydrogen	0	1286
helium	0	971
nitrogen	0	334
oxygen	0	315
air	0	332
air	20	344
glycerin	20	1920
water	20	1484
mercury	20	1420
benzol	20	1320
ethanol	20	1170
diamond	20	17000
aluminum	20	5110
glass	20	4000–6000

Table 3.2. Speed of sound in various media at standard pressure

- Typical wavelength
  - $c_\kappa = 1000\ \mathrm{m\ s^{-1}}$
  - $\nu = 1\ \mathrm{kHz}$
  - $\lambda = c_\kappa / \nu = \mathbf{1\ m}$

3.3.5. Pressure Waves

optional

- **Characteristic numbers**
  - Strouhal number

$$Sr = \frac{t\bar{v}}{l}$$

  - Mach number

$$Ma = \frac{\bar{v}}{c_\kappa}$$

- Intensity
  - ➢ Power per
  - ➢ Surface area

$$I = \varrho_N \frac{1}{2} m v_\xi^2 c_\kappa = \frac{1}{2} \varrho c_\kappa v_\xi^2$$

- Radiation pressure

$$p_{\text{rad}} = \varrho_N \frac{1}{2c_\kappa} m v_\xi^2 c_\kappa = \frac{I}{c_\kappa}$$

3.3.5. Damping of Pressure Waves

optional

- Energy dissipation
  - Inner friction of fluid
  - Heating of fluid

- Friction term in wave equation

$$\frac{\partial^2 \xi}{\partial t^2} = c_\kappa^2 \frac{\partial^2 \xi}{\partial z^2} + \gamma_{\text{damp}} \frac{\partial^3 \xi}{\partial z^2 \partial t}$$

- Planar wave

$$I(z) = I_0 \mathrm{e}^{-\Gamma z}$$

$$\Gamma = 2\pi^2 \nu^2 \gamma_{\text{damp}} / c_\kappa^3$$

3.3. Fluid Dynamics

optional

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3.3.6. Flow through Constriction

optional

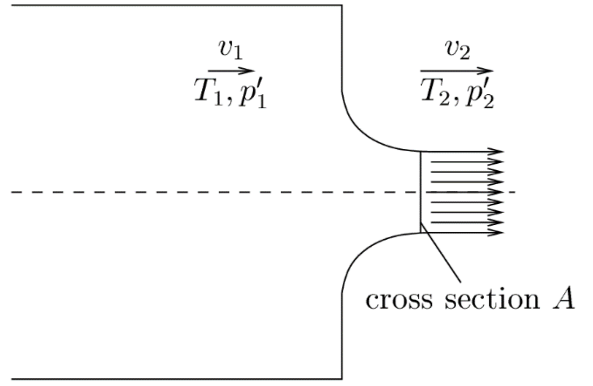


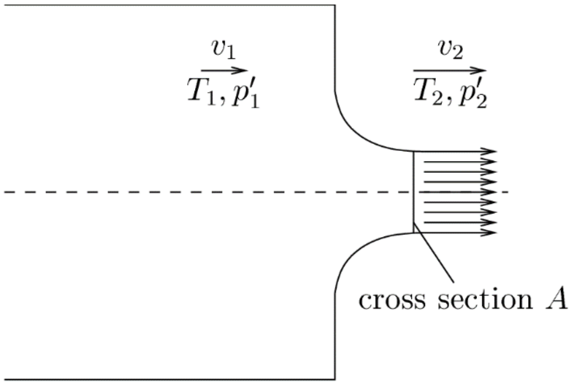
Fig. 3.24. Outflow through a nozzle from a pressurized chamber under a differential pressure $p'_1 - p'_2$

- Outflow
  - Pressurized chamber
  - Nozzle
  - Flow velocities v_1 and v_2 inside and outside chamber

- Mass flow rate I_m
  - Function of pressure ratio $p'_2/p_1{}'$
  - Shape of constriction

3.3.6. Flow through Constriction

optional



- Bendemann formula
  - Flow rate I_m
  - Volume per mass V_m
  - Isentropic conditions
  - $v_1 \ll v_2$
- Outlet function
  - Dimensionless
  - Isentropic coefficient $\gamma \approx \frac{5}{3}$

$$I_{m,\mathrm{theor}} = A\Psi_{A,2}\sqrt{2\frac{p'_1}{V_{m,1}}}$$

$$\Psi_{A,2} = \sqrt{\frac{\gamma}{\gamma-1}\left[\left(\frac{p'_2}{p'_1}\right)^{\frac{2}{\gamma}} - \left(\frac{p'_2}{p'_1}\right)^{\frac{\gamma+1}{\gamma}}\right]}$$

3.3.6. Laval Pressure Ratio

optional

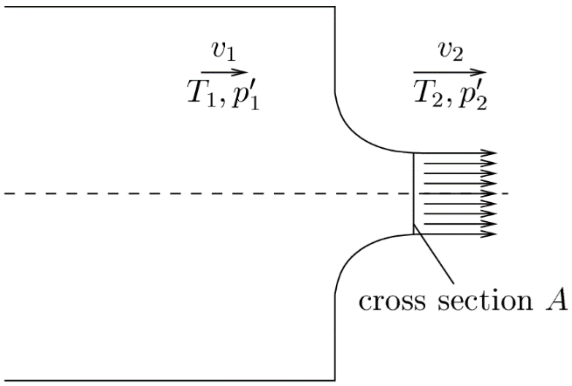
- Outlet function
  - Maximum value

$$\Psi_{A,\max} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \sqrt{\frac{\gamma}{\gamma+1}}$$

  - Obtained at critical Laval ratio

$$\tilde{p}_{\mathrm{Laval}} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

    - Propagation of pressure signal limited by speed of sound



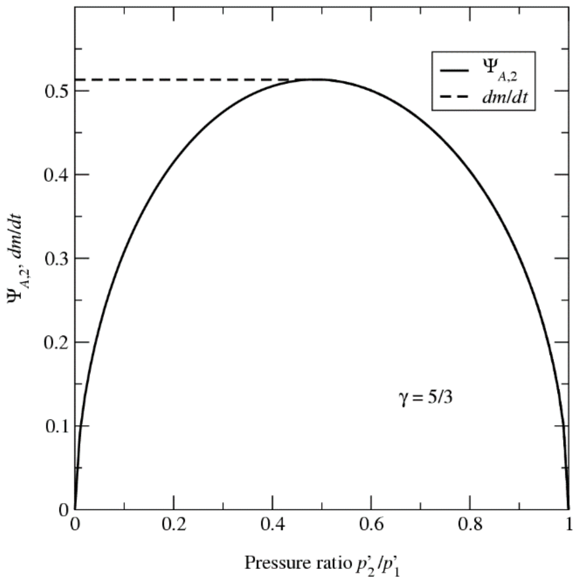


Fig. 3.24. Outlet function $\Psi_{A,2}$ and mass flow dm/dt as a function of the pressure ratio p'_2/p'_1 for an isentropic coefficient $\gamma = 5/3$

3.3.6. Outflow of Compressible Gases

optional

- Additional correction factors
  - Velocity factor ϕ_1
    - Outlet shape
    - Velocity
  - Contraction number ϕ_2
    - Outlet shape

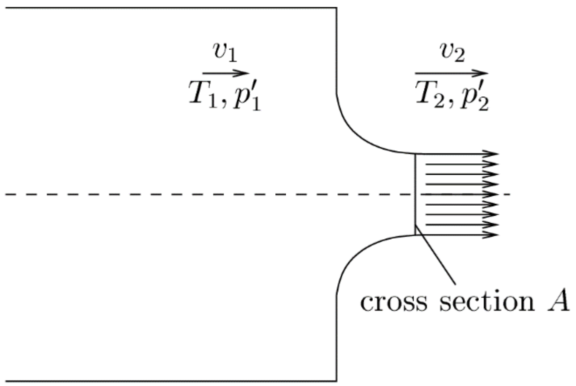


Fig. 3.24. Outflow through a nozzle from a pressurized chamber under a differential pressure $p'_1 - p'_2$

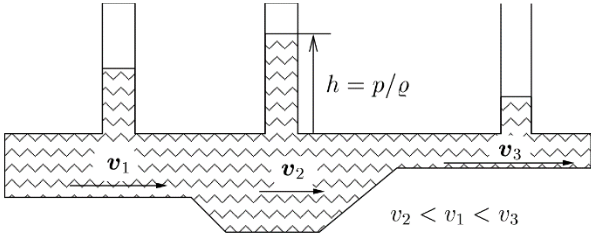
outlet geometry	velocity factor ϕ_1	contraction number ϕ_2	outlet number ϕ_{outlet}
sharp edges	0.97	$0.61 \ldots 0.64$	$0.59 \ldots 0.62$
rounded	$0.97 \ldots 0.99$	1	$0.97 \ldots 0.99$

Table 3.4. Typical values for the velocity factor ϕ_1, contraction number ϕ_2 and overall outlet coefficient ϕ_{outlet} as determined by experiments with different outlet shapes

Summary

Bernoulli equation

$$p + \frac{\varrho}{2} v^2 = p_{\mathrm{tot}} = \mathrm{const.}$$



Cavitation Coanda effect

Drag force (on body)

$$F = C_{\mathrm{d}} \frac{\varrho}{2} v^2 A_{\mathrm{d}}$$

Drag coefficient

$$C_{\mathrm{d}} = \frac{12}{Re}$$

Stokes drag (for sphere)

$$F_{\mathrm{Stokes}} = -6\pi\eta v r_0 = -\frac{6 A_{\mathrm{d}} \varrho v^2}{Re}$$